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A Mathematical Model of the Boomerang

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Abstract

In this paper we propose the mathematical model of a boomerang, flying in undisturbed air. Formulation of aerodynamic forces and moments is based on the strip theory, which proved to be quite efficient in problems of dynamics of a rigid body moving in the medium. Equations of motion of the boomerang are derived and investigated numerically. Trajectories of the boomerang obtained in numerical experiment are very similar to the trajectories observed in reality.

Keywords:
Boomerang, Aerodynmics, Differential equations, Numerical Analysis

1. Formulatiom of a Mathematical Model

1.1. Coordinate Frames. Kinematical Equations

The model of a boomerang, considered in this paper (Fig. 1) differs from the traditional models (see e.g. [1, 2]). The basic theoretical background used in the formulation of the model have been taken from the theory of small wind power generators developing by V.A. Samsonov and his co-authors [3]. The model has four mutually perpendicular wings (shaded in black). Wings are joined by the holders. We suppose that the construction of the boomerang is completely symmetric. Therefore point O is the center of mass of the boomerang.

Let us introduce the fixed coordinate system O_*XYZ with the origin at the point O_* – the center of mass of the boomerang at start. The O_*X - axis is directed upward, and the direction of the O_*Y - axis is chosen so that the O_*XY - plane is the plane of throwing. Therefore at the initial instant the vector $\mathbf{V}$ of velocity of the center of mass is in the plane O_*XY. We denote the unit vectors of this coordinate system by $\mathbf{e}_X$, $\mathbf{e}_Y$, $\mathbf{e}_Z$ respectively.

We introduce also the coordinate system $O\xi\eta\zeta$, where $O\zeta$ is the axis of symmetry of the boomerang. Let $\mathbf{e}_{xi}$, $\mathbf{e}_{eta}$, $\mathbf{e}_{zeta}$ be the unit vectors of this coordinate system. We define the mutual

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orientation of the systems O_*XYZ and $O\xi\eta\zeta$ by the Euler angles θ, σ and ψ as follows:

$$\left\{\begin{array}{c} X \\ Y \\ Z \end{array}\right\} = \left(\begin{array}{ccc} p_1 & p_2 & p_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \delta_1 & \delta_2 & \delta_3 \end{array}\right) \left\{\begin{array}{c} \xi \\ \eta \\ \zeta \end{array}\right\},$$

$$\left(\begin{array}{ccc} p_1 & p_2 & p_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \delta_1 & \delta_2 & \delta_3 \end{array}\right) = \left(\begin{array}{ccc} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{array}\right) \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\sigma & -\sin\sigma \\ 0 & \sin\sigma & \cos\sigma \end{array}\right) \left(\begin{array}{ccc} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{array}\right)$$

Thus, for the components of the angular velocity $\boldsymbol{\omega}$ of the system $O\xi\eta\zeta$ we have the following expressions:

$$\boldsymbol{\omega} = \omega_\xi \mathbf{e}_\xi + \omega_\eta \mathbf{e}_\eta + \omega_\zeta \mathbf{e}_\zeta,$$

$$\omega_\xi = \dot{\theta}\sin\psi\cos\sigma + \dot{\sigma}\cos\psi,\ \ \omega_\eta = \dot{\theta}\cos\psi\cos\sigma - \dot{\sigma}\sin\psi,\ \ \omega_\zeta = \dot{\psi} - \dot{\theta}\sin\sigma.$$

Finally we introduce the coordinate system $Oxyz$ which is rigidly connected with the boomerang. The Oz - axis of this system coincides with the axis $O\zeta$. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the unit vectors of this coordinate system. We denote the angle between $O\xi$ and Ox by γ.

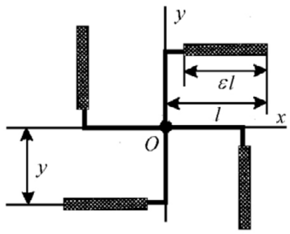

Figure 1: The Model of Boomerang

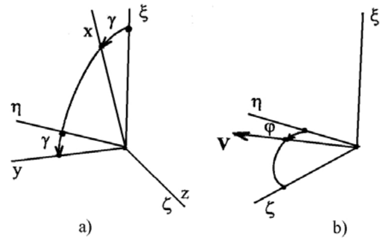

Figure 2: Orientation of the Coordinate Systems

Now let us define more precisely the motion of the coordinate system $O\xi\eta\zeta$. The position of the $O\zeta$ - axis is completely defined by the position of the axis of symmetry of the boomerang. We suppose the other two axes $O\xi$ and $O\eta$ are directed such that the vector $\mathbf{V}$ of velocity of the center of mass is in the $O\eta\zeta$ - plane during the motion. We denote the angle between $\mathbf{V}$ and the $O\eta$ - axis by φ (Fig. 2). Therefore, we have the following kinematic equations for the absolute coordinates of the center of mass of the boomerang:

$$\dot{X} = V\left(p_2\cos\varphi + p_3\sin\varphi\right),\ \dot{Y} = V\left(\lambda_2\cos\varphi + p_3\sin\varphi\right),\ \dot{Z} = V\left(\delta_2\cos\varphi + \delta_3\sin\varphi\right). \tag{1}$$

1.2. Dynamical Equations

Let us consider the free motion of a boomerang in an unperturbed air. We suppose that the boomerang is subjected to the gravity and the distributed system of aerodynamic forces and moments.

We will write the dynamical equations of motion of the boomerang with respect to the system $O\xi\eta\zeta$. By the impulse-momentum change theorem we get:

$$m\left(\dot{\mathbf{V}} + \boldsymbol{\omega}\times\mathbf{V}\right) = \mathbf{F} + m\mathbf{g}. \tag{2}$$

Here $m\mathbf{g}$ is the gravity force,

$$\mathbf{F} = F_\xi \mathbf{e}_\xi + F_\eta \mathbf{e}_\eta + F_\zeta \mathbf{e}_\zeta$$

is the resulting vector of aerodynamic forces,

$$\boldsymbol{\omega} = \omega_\xi \mathbf{e}_\xi + \omega_\eta \mathbf{e}_\eta + \omega_\zeta \mathbf{e}_\zeta, \ \mathbf{V} = V\left(\mathbf{e}_\eta \cos\varphi + \mathbf{e}_\zeta \sin\varphi\right),$$

$$\dot{\mathbf{V}} = \dot{V}\left(\mathbf{e}_\eta \cos\varphi + \mathbf{e}_\zeta \sin\varphi\right) + V\dot{\varphi}\left(-\mathbf{e}_\eta \sin\varphi + \mathbf{e}_\zeta \cos\varphi\right).$$

We can rewrite equation (2) in a scalar form:

$$\begin{aligned} & mV\left(\omega_\eta \sin\varphi - \omega_\zeta \cos\varphi\right) = F_\xi - mgp_1, \\ & m\left[\dot{V}\cos\varphi - V\left(\dot{\varphi} + \omega_\xi\right)\sin\varphi\right] = F_\eta - mgp_2, \\ & m\left[\dot{V}\sin\varphi + V\left(\dot{\varphi} + \omega_\xi\right)\cos\varphi\right] = F_\zeta - mgp_3. \end{aligned} \tag{3}$$

Solving system (3) with respect to $\dot{V}$ and $\dot{\varphi}$ we get:

$$\begin{aligned} & \dot{V} = \left(F_\eta \cos\varphi + F_\zeta \sin\varphi\right) - g\left(p_2 \cos\varphi + p_3 \sin\varphi\right), \\ & \dot{\varphi} = -\omega_\xi + \left(F_\zeta \cos\varphi - F_\eta \sin\varphi\right) / (mV) + g\left(p_2 \sin\varphi + p_3 \cos\varphi\right) / V, \\ & \omega_\zeta \cos\varphi = \omega_\eta \sin\varphi - F_\xi / (mV) + gp_1 / V. \end{aligned} \tag{4}$$

Similarly, by the theorem on the change of moment of momentum we have:

$$\dot{\mathbf{G}} + \omega \times \mathbf{G} = \mathbf{M}, \tag{5}$$

where $\mathbf{M} = M_\xi \mathbf{e}_\xi + M_\eta \mathbf{e}_\eta + M_\zeta \mathbf{e}_\zeta$ is the central aerodynamic moment,

$$\mathbf{G} = A\omega_\xi \mathbf{e}_\xi + A\omega_\eta \mathbf{e}_\eta + C\Omega \mathbf{e}_\zeta, \ \dot{\mathbf{G}} = A\dot{\omega}_\xi \mathbf{e}_\xi + A\dot{\omega}_\eta \mathbf{e}_\eta + C\dot{\Omega} \mathbf{e}_\zeta, \ \Omega = \omega_\zeta + \dot{\gamma}.$$

In the scalar form equation (5) can be written as follows:

$$A\dot{\omega}_\xi + \omega_\eta\left(C\Omega - A\omega_\zeta\right) = M_\xi, \ A\dot{\omega}_\eta + \omega_\xi\left(A\omega_\zeta - C\Omega\right) = M_\eta, \ C\dot{\Omega} = M_\zeta. \tag{6}$$

To complete the system of equations (1), (4), (6) we need to obtain the expressions for aerodynamic force and moment.

1.3. Mathematical Model for the Force and Moment

Let us briefly describe the method for obtaining the expressions for aerodynamic force and moment acting on the boomerang. We suppose that the aerodynamic forces acting on the boomerang are concentrated only on its wings. Assuming that the wings are similar one to another we can calculate the corresponding aerodynamic force and moment only for one wing. We choose one of the four wings and mentally cut it out by two planes perpendicular to the longitudinal line of the wing. As a result we obtain an elementary part (element s) of the wing with an infinitesimal thickness. We will find the aerodynamic force and moment acting on this element. The total aerodynamic influence on the wing can be obtained by integration of the aerodynamic forces and moments acting on every element of the wing.

Let us assume that aerodynamic force and moment acting on the element s are situated in the plane of this element. They depend only on $\mathbf{V}_{ef}$ - the projection of velocity of the center of pressure of the given element onto the plane of this element.

We choose the center of pressure of the considered element s as a reduction center for the forces acting on this element. Let us denote by $\mathbf{r}$ the radius-vector of the center of pressure of the given element with respect to the center of mass O. The resulting aerodynamic force acting on the element s can be represented as a sum of the drag force and the lift force:

$$\Delta\mathbf{F} = \Delta\mathbf{W} + \Delta\mathbf{L}.$$

The corresponding aerodynamic moment has the form:

$$\Delta\mathbf{M} = \mathbf{r} \times \Delta\mathbf{F}.$$

We can write expressions $\Delta\mathbf{W}$ and $\Delta\mathbf{L}$ as follows:

$$\Delta\mathbf{W} = -0.5c_x(\alpha)\rho\Delta S\, V_{ef}\mathbf{V}_{ef}, \quad \Delta\mathbf{L}\cos\alpha = 0.5c_y(\alpha)\rho\Delta S\left(\mathbf{V}_{ef}\times\mathbf{n}\right)\times\mathbf{V}_{ef}. \tag{7}$$

Here ρ is an atmosphere's density, ΔS is the area of the element s, $c_x(\alpha)$ and $c_y(\alpha)$ are dimensionless aerodynamic coefficients. They depend only on the angle of attack α - the angle between $\mathbf{V}_{ef}$ and the plane Π passing through the longitudinal line of the wing and its chord. Let $\mathbf{n}$ be the normal vector to this plane.

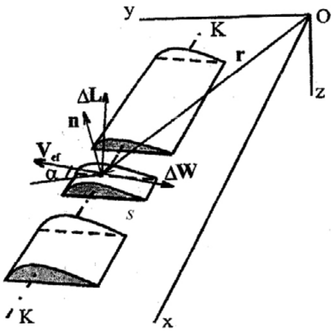

Figure 3: The wing of a boomerang and aerodynamic forces acting on it

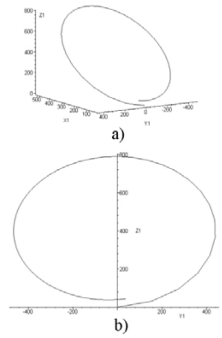

Figure 4: Trajectory of the center of mass of a boomerang

In order to calculate the aerodynamic forces $\Delta\mathbf{W}$ and $\Delta\mathbf{L}$ we need to determine the angle of attack α for every element. We suppose that the cross-section of the wing doesn't change its form along the wing-span. This means that the centers of pressure of all elements s of the wing are situated on the straight line K. We consider the wing of the boomerang which is in the first quadrant of the plane Oxy. Then the Ox - axis has the same direction as the corresponding line K of this wing. We denote the coordinates of the center of pressure of the given element s by x and y respectively. Then we can conclude that the y - coordinate is the fixed value while the x - coordinate is changed from $l(1-\varepsilon)$ to l (see Fig. 1).

As a result we have the following expressions for the radius-vector $\mathbf{r}$ and the velocity $\mathbf{V}_{ef}$:

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2,$$
$$\mathbf{V}_{ef} = (V\cos\varphi\cos\gamma + \Omega x)\,\mathbf{e}_2 + \left[V\sin\varphi + \left(x\omega_\xi + y\omega_\eta\right)\sin\gamma + \left(y\omega_\xi - x\omega_\eta\right)\cos\gamma\right]\mathbf{e}_3,$$
$$V_{ef} = \left[(V\cos\varphi\cos\gamma + \Omega x)^2 + \left[V\sin\varphi + \left(x\omega_\xi + y\omega_\eta\right)\sin\gamma + \left(y\omega_\xi - x\omega_\eta\right)\cos\gamma\right]^2\right]^{1/2}.$$

Since the angle α is changed during the boomerang flight we need to express it through the other variables and parameters of the problem. For this purpose we introduce the fixed angle β – the angle between the plane Π and the equatorial plane (Fig. 3). Then we can write the vectors $\mathbf{n}$ and $\mathbf{n}_*$ as follows:

$$\mathbf{n} = -\mathbf{e}_2 \sin\beta + \mathbf{e}_3 \cos\beta, \quad \mathbf{n}_* = \mathbf{e}_2 \cos\beta + \mathbf{e}_3 \sin\beta.$$

Calculating the corresponding scalar and vector products $\left(\mathbf{V}_{ef} \cdot \mathbf{n}\right)$, $\mathbf{V}_{ef} \times \mathbf{n}$, $\mathbf{V}_{ef} \times \mathbf{n}_*$ we obtain the set of relations to determine the angle α:

$$V_{ef} \sin\alpha = (V\cos\varphi\cos\gamma + \Omega x)\sin\beta - \left[V\sin\varphi + \left(x\omega_\xi + y\omega_\eta\right)\sin\gamma + \left(y\omega_\xi - x\omega_\eta\right)\cos\gamma\right]\cos\beta,$$
$$V_{ef} \cos\alpha = (V\cos\varphi\cos\gamma + \Omega x)\cos\beta + \left[V\sin\varphi + \left(x\omega_\xi + y\omega_\eta\right)\sin\gamma + \left(y\omega_\xi - x\omega_\eta\right)\cos\gamma\right]\sin\beta.$$

We will write equations of motion of the boomerang in the dimensionless form. Let us introduce the dimensionless variables:

$$\omega_1 = \omega_\xi l/V_0^2,\ \omega_2 = \omega_\eta l/V_0^2,\ \omega_3 = \omega_\zeta l/V_0^2,\ \Omega_3 = \Omega l/V_0^2,\ u = V/V_0,$$
$$X_1 = X/l,\ Y_1 = Y/l,\ Z_1 = Z/l,\ x_1 = x/l$$

and the dimensionless time:

$$\tau = V_0 t/l.$$

We introduce also the following parameters:

$$e = y/l,\ \mu = \rho S l/m,\ \chi = ml^2/A,\ k = gl/V_0^2,\ J = C/A.$$

The equation for the angle α can be written as follows:

$$\sin\alpha = \frac{\left[(u\cos\varphi\cos\gamma + \Omega_3 x_1)\sin\beta - (u\sin\varphi + (\omega_1 x_1 + \omega_2 e)\sin\gamma + (\omega_1 e - \omega_2 x_1)\cos\gamma)\cos\beta\right]}{\sqrt{(u\cos\varphi\cos\gamma + \Omega_3 x_1)^2 + (u\sin\varphi + (\omega_1 x_1 + \omega_2 e)\sin\gamma + (\omega_1 e - \omega_2 x_1)\cos\gamma)^2}}.$$

Further we will consider the angle α as a small angle. To provide this condition we will suppose that the variables φ, ω_1, ω_2 and the angle β are also small. Under these conditions we obtain the approximate formula for α:

$$\alpha = \beta - \frac{u\varphi + (\omega_1 x_1 + \omega_2 e)\sin\gamma + (\omega_1 e - \omega_2 x_1)\cos\gamma}{u\cos\gamma + \Omega_3 x_1}$$

and also

$$c_x(\alpha) = c_{x0} + c_{x1}\alpha^2, \quad c_y(\alpha) = c_{y1}\alpha.$$

We substitute these approximate expressions into the formulae (7). Then we integrate the obtained expressions for $\Delta\mathbf{W}$ and $\Delta\mathbf{L}$ by x_1 from $1-\varepsilon$ to 1. Note that the variables u, φ, ω_1, ω_2, Ω_3 are changed slowly during the whole period of changing of the angle $\gamma : [0, 2\pi]$. Therefore we can average the obtained expressions with respect to γ. The resulting components of the aerodynamic force and moment have a very complicated form and we omit them here. Finally equations of motion of our model of a boomerang can be written as follows:

$$\begin{aligned}\frac{du}{d\tau} &= 2\mu\varepsilon c_{x1} u^2 \beta\varphi - \mu e\varepsilon\left(2c_{x0} - c_{y1} + 2c_{x1}\right)u\varphi\omega_1 - \mu\varepsilon(2-\varepsilon)\left(c_{x0} + c_{x1}\beta^2 + c_{y1}\varphi^2\right)u\Omega_3 + \\ &+ \frac{\mu}{2}\varepsilon(2-\varepsilon)\left(2c_{x0} - c_{y1} + 2c_{x1}\right)u\varphi\omega_2 + \frac{\mu}{2}e\varepsilon(2-\varepsilon)\left(2c_{x1} - c_{y1}\right)\Omega_3\beta\omega_1 + \\ &+ \frac{2\mu}{3}c_{y1}\varepsilon\left(\varepsilon^2 - 3\varepsilon + 3\right)\Omega_3^2\beta\varphi - \frac{\mu}{3}\varepsilon\left(\varepsilon^2 - 3\varepsilon + 3\right)\left(2c_{x1} - c_{y1}\right)\Omega_3\beta\omega_2 - k\left(p_3\varphi + p_2\left(1 - \frac{\varphi^2}{2}\right)\right),\end{aligned}$$

$$\frac{d\varphi}{d\tau} = -\omega_1 + \mu c_{y1}\varepsilon u\beta - \mu e\varepsilon\left(c_{x0} + c_{y1}\right)\omega_1 - \mu\varepsilon c_{y1}\left(2-\varepsilon\right)\Omega_3\varphi +$$
$$+\frac{\mu}{2}\varepsilon\left(2-\varepsilon\right)\left(c_{x0} + c_{y1}\right)\omega_2 + \frac{2\mu}{3}c_{y1}\varepsilon\left(\varepsilon^2 - 3\varepsilon + 3\right)\frac{\Omega_3^2}{u}\beta + \frac{k}{u}\left(p_2\varphi - p_3\left(1 - \frac{\varphi^2}{2}\right)\right),$$

$$\dot{X}_1 = u\left(p_3\varphi + p_2\left(1 - \frac{\varphi^2}{2}\right)\right), \quad \dot{Y}_1 = u\left(\lambda_3\varphi + \lambda_2\left(1 - \frac{\varphi^2}{2}\right)\right), \quad \dot{Z}_1 = u\left(\delta_3\varphi + \delta_2\left(1 - \frac{\varphi^2}{2}\right)\right),$$

$$\frac{d\omega_1}{d\tau} + J\omega_2\Omega_3 + \mu\chi e\varepsilon\left(c_{x0} + c_{y1}\right)u^2\varphi - \mu\chi e\varepsilon\left(2-\varepsilon\right)c_{y1}u\Omega_3\beta -$$
$$-\frac{k\omega_2 p_1}{u} + \frac{\mu\chi}{4}\varepsilon\left(2-\varepsilon\right)\left(\varepsilon^2 - 2\varepsilon + 2 + 2e^2\right)\left(c_{x0} + c_{y1}\right)\Omega_3\omega_1,$$

$$\frac{\omega_2}{d\tau} - J\omega_1\Omega_3 - \frac{\mu\chi}{2}\varepsilon\left(2-\varepsilon\right)\left(c_{x0} + c_{y1}\right)u^2\varphi + \frac{2\mu\chi}{3}\varepsilon c_{y1}\left(\varepsilon^2 - 3\varepsilon + 3\right)u\Omega_3\beta +$$
$$+\frac{k\omega_1 p_1}{u} + \frac{\mu\chi}{4}\varepsilon\left(2-\varepsilon\right)\left(\varepsilon^2 - 2\varepsilon + 2 + 2e^2\right)\left(c_{x0} + c_{y1}\right)\Omega_3\omega_2,$$

$$\frac{d\Omega_3}{d\tau} = -\frac{\mu\chi}{8J}\varepsilon\left(2-\varepsilon\right)\left(\varepsilon^2 - 2\varepsilon + 2 + 2e^2\right)\left(c_{x0} - 2c_{y1} + 2c_{x1}\right)\left(\omega_1^2 + \omega_2^2\right) +$$
$$+\frac{\mu\chi}{2J}e\varepsilon\left(2-\varepsilon\right)\left(2c_{x1} - c_{y1}\right)u\beta\omega_1 - \frac{\mu\chi}{3J}\varepsilon\left(\varepsilon^2 - 3\varepsilon + 3\right)\left(2c_{x1} - c_{y1}\right)u\beta\omega_2 +$$
$$+\frac{2\mu\chi}{3J}\varepsilon\left(\varepsilon^2 - 3\varepsilon + 3\right)\left(2c_{x1} - c_{y1}\right)\Omega_3 u\beta\varphi - \frac{\mu\chi}{2J}\varepsilon\left(2-\varepsilon\right)\left(\varepsilon^2 - 2\varepsilon + 2\right)\left(c_{x0} + c_{x1}\beta^2\right)\Omega_3^2 -$$
$$-\frac{\mu\chi}{2J}\varepsilon\left(2-\varepsilon\right)\left(c_{x0} - 2c_{y1}\varphi^2 + c_{x1}\beta^2 + 2c_{x1}\varphi^2\right)u^2.$$

To complete this system of equations we need to add the kinematic equations:

$$\frac{d\theta}{d\tau} = \frac{\sin\psi}{\cos\sigma}\omega_1 + \frac{\cos\psi}{\cos\sigma}\omega_2, \; \frac{d\sigma}{d\tau} = \omega_1\cos\psi - \omega_2\sin\psi, \; \frac{d\psi}{d\tau} = \frac{\sin\psi\sin\sigma}{\cos\sigma}\omega_1 + \frac{\cos\psi\sin\sigma}{\cos\sigma}\omega_2 + \omega_3$$

and the expressions for p_i, λ_i, δ_i, $i = 1, 2, 3$.

2. Numerical Simulation

The obtained system of equations has been investigated numerically at the following values of parameters:

$$\chi = 1.7, \; J = 2, \; k = 0.00004, \; e = 0.1, \; \mu = 0.1, \; \beta = 0.034, \; \varepsilon = 0.2, \; c_{x0} = 0.01, \; c_{x1} = 4, \; c_{y1} = 5.$$

The initial conditions have been chosen as follows:

$$u(0) = 1, \; \omega_1(0) = 0, \; \omega_2(0) = 0, \; \Omega_3(0) = 1.5, \; X_1(0) = 20, \; Y_1(0) = 0, \; Z_1(0) = 0,$$
$$\varphi(0) = 0, \; \psi(0) = 0, \; \sigma(0) = 0, \; \theta(0) = \pi/6.$$

Numerical simulation, performed with the given values of parameters and initial conditions shows that the center of mass of the boomerang describes the almost periodic trajectory (see Fig. 4). This trajectory is very similar to trajectories of an actual boomerang. Therefore we can conclude that the proposed mathematical model of a boomerang describes effectively the behavior of a boomerang in flight.

Acknowledgements

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